Name:	
Class:	

JAMES RUSE AHS
GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2013

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged working.

Total Marks 70

Section I: **10 marks**

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: **60 Marks**

- Attempt Question 11 - 14
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hour & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

SECTION I MULTIPLE CHOICE (10 marks)

Attempt Question 1 – 10 (1 mark each)
Allow approximately 15 minutes for this section

1. An object is projected with a velocity of 40 ms^{-1} at an angle of $\tan^{-1}\frac{8}{6}$ to the horizontal. If air resistance is neglected and acceleration due to gravity is taken as 10 ms^{-2}, what is the initial vertical component of its velocity?

(A) 32 ms^{-1}

(B) 24 ms^{-1}

(C) $40\tan\frac{8}{6}$ ms^{-1}

(D) $40\sin\frac{8}{10}$ ms^{-1}

2. For the function $f(x) = 3\sin^{-1}(\frac{x}{4})$, the domain and range of $y = f(x)$ (where x and y are real numbers) are:

(A) $-4 \le x \le 4$; $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(B) $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$; $-4 \le y \le 4$

(C) $-3 \le x \le 3$; $-2\pi \le y \le 2\pi$

(D) $-2\pi \le x \le 2\pi$; $-3 \le y \le 3$

3. A particle moves in a straight line. At time t seconds, where $t \ge 0$, its displacement x metres from the origin and its velocity v are such that $v = 25 + x^2$. If initially the particle is 5m to the right of the origin, then t is equal to:

(A) $\tan^{-1}(\frac{x}{5}) - \frac{\pi}{4}$

(B) $\frac{1}{5}\tan^{-1}(\frac{x}{5}) - \frac{\pi}{20}$

(C) $25x + \frac{x^3}{3}$

(D) $25x + \frac{x^3}{3} + \frac{500}{3}$

4. What are appropriate values of R and θ such that $\sqrt{3}cosx + sinx \equiv Rcos(x + \theta)$

(A) $R = \sqrt{2}$
$\theta = \dfrac{\pi}{6}$

(B) $R = \sqrt{2}$
$\theta = \dfrac{11\pi}{6}$

(C) $R = 2$
$\theta = \dfrac{\pi}{6}$

(D) $R = 2$
$\theta = \dfrac{11\pi}{6}$

5. If $cos2x < -sinx$ for $0 \leq x \leq \pi$, then:

(A) This is true for $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

(B) There are no x values in this domain for which this is true

(C) This is true for all x values within this domain except $x = \frac{\pi}{2}$

(D) This is true for all x values within this domain

6. $(x - a)$ is a factor of the polynomial $P(x)$, where a is an integer.
If $P(x) = x^3 - kx^2 + 2kx - 8$, the values of k for which $P(x)$ has real roots are:

(A) $-6 \leq k \leq 2$

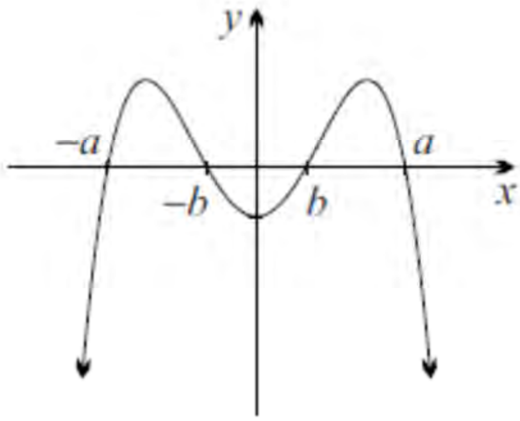
(B) $-2 \leq k \leq 6$

(C) $k \leq -6$ or $k \geq 2$

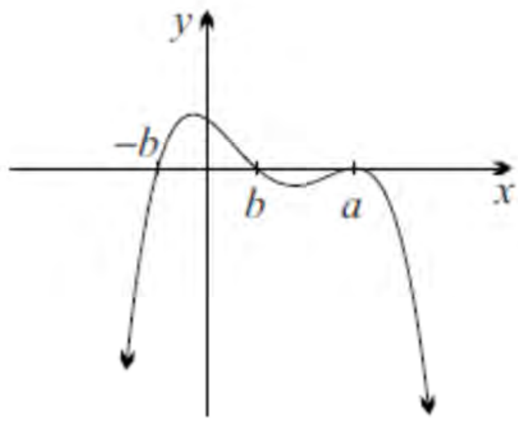
(D) $k \leq -2$ or $k \geq 6$

7. Which diagram best represents $y = P(x)$ if $P(x) = (x - a)^2(b^2 - x^2)$, and $a > b$?

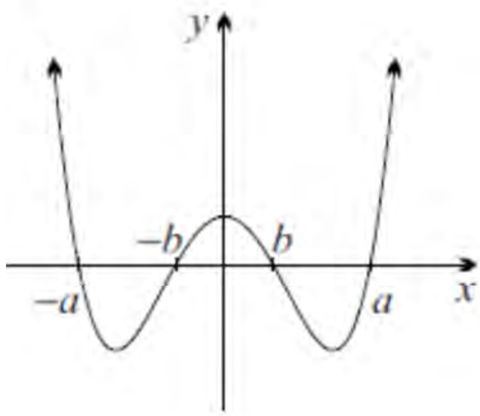
(A)



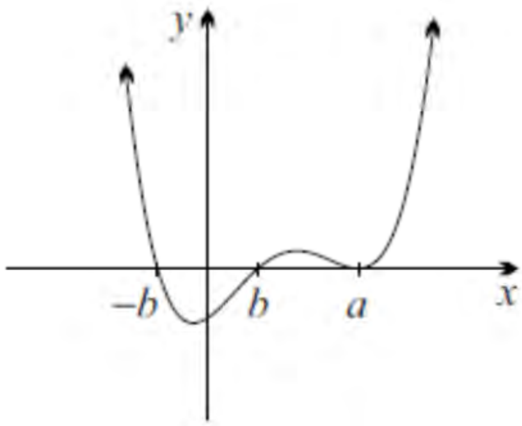
(C)



(B)



(D)



8. If $cos\theta = -\frac{3}{5}$ and $0 < \theta < \pi$, then $\tan\frac{\theta}{2}$ is equal to:

(A) $-\frac{1}{3}$ or -3

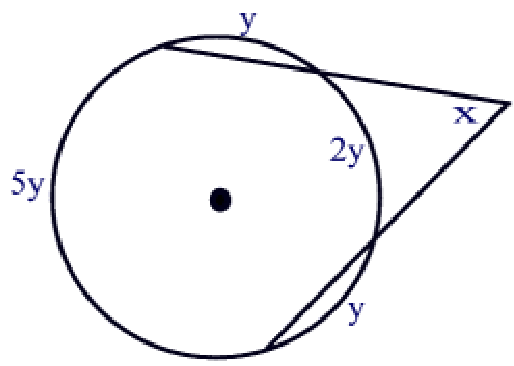
(B) $\frac{1}{3}$ or 3

(C) -2

(D) 2

9. Two secants of equal length intersect at an external point. Given the arc lengths shown in the diagram, the number of degrees in the angle labeled x is:

DIAGRAM NOT TO SCALE



(A) 40

(B) 60

(C) 80

(D) 140

10. Two year 12 students are to be randomly selected from a pool of N year 12 students, n of whom are from James Ruse. If it is known that at least one student is from James Ruse, what is the chance that both students are from James Ruse?

(A) $\frac{n-1}{2N-n-1}$

(B) $\frac{n-1}{2N+n+1}$

(C) $\frac{n-1}{2N+n-1}$

(D) $\frac{n-1}{2N-n+1}$

END OF SECTION I

SECTION II EXTENDED RESPONSE (60 marks)

Total Marks is 60
Attempt Question 11 – 14.
Allow approximately 1 hour & 45 minutes for this section.

*Answer all questions, starting each new question on a new sheet of paper with your **student ID number** in the top right hand corner and the question number on the left hand side of your paper. All necessary working must be shown in each and every question.*

QUESTION 11 *(15 Marks)* START A NEW PAGE **Marks**

(a) The functions $y = 4x^3 + 3x - 1$ and $y = 1 - \ln(2x)$ intersect at $(\frac{1}{2}, 1)$. Find the acute angle between the tangents of the two curves at the point of intersection. Give your answer to the nearest minute. 3

(b) (i) State the range of $y = \tan^{-1}\frac{\sqrt{x^2-4}}{2}$ 1

(ii) Find $\frac{dy}{dx}$ for the function $y = \tan^{-1}\frac{\sqrt{x^2-4}}{2}$ 3

(c) The concentration, C, of good bacteria in a healthy intestine is usually 110 bacteria per gram of intestinal contents. After a bout of food poisoning, this concentration drops to 15 bacteria per gram. The rate of increase of the concentration of good bacteria is proportional to the difference from the normal 110 bacteria per gram, that is,

$$\frac{dC}{dt} = k(110 - C)$$

where t is the number of hours since suffering food poisoning.

(i) Show that $C = 110 - Ae^{-kt}$ satisfies the above equation. 1

(ii) 19 hours after suffering food poisoning, Harold has a concentration of 20 bacteria per gram. How long after suffering food poisoning will the concentration of good bacteria in his intestines reach 90% of its healthy state? Give your answer to the nearest hour. 3

(d) PQ is the chord of contact of the parabola $x^2 = 4y$ from the external point $A(x_1, y_1)$ with equation $xx_1 = 2a(y + y_1)$.

(i) Show that the midpoint, M, of PQ has the coordinates $(x_1, \frac{x_1^2}{2} - y_1)$. 2

(ii) If A moves along the line $3x - y - 1 = 0$, show that the equation of the locus of M is a parabola of the form $(x - h)^2 = \pm 4a(y - k)$. 2

QUESTION 12 *(15 Marks)* START A NEW PAGE **Marks**

(a) Find $\int \frac{x^2}{(a^2-x^2)^{\frac{3}{2}}}\,dx$ by making the substitution $x = a\sin u$. 3

(b) Assume that, for all real numbers x and all positive integers n, $(1+x)^n$ can be written as

$$\sum_{r=0}^{n} {}^{n}C_r\, x^r$$

Find a simple expression for: 2

$$\sum_{r=0}^{n} r.\ {}^{n}C_r$$

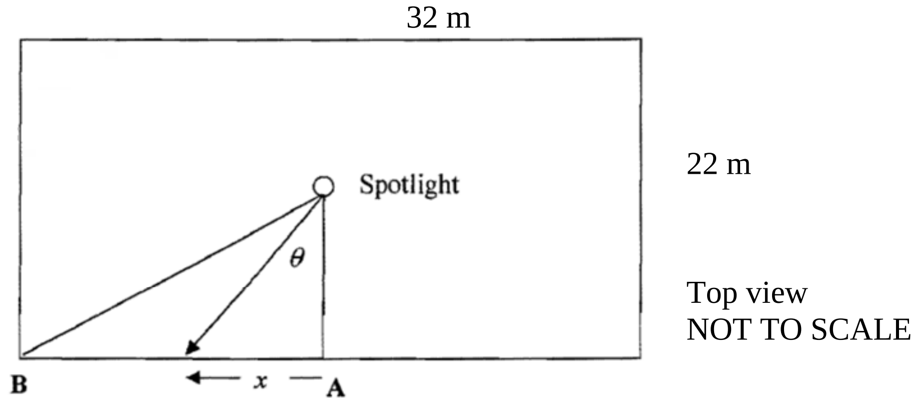
(c) The water level at a harbour entrance approximates to simple harmonic motion. On a particular day high tide occurred at 6.00 am. There was a depth of 10 m of water at that time. At low tide, which occurred at 11.50 am, there was a depth of 3 m of water.

(i) Find, to the nearest cm, the depth of water at the harbour entrance at 2 am earlier that same day. 3

(ii) Write an expression, in exact form, for the general solution for the times, t, when the depth of water at the harbour entrance is 7.5 m. 1

(iii) What is the latest time before midnight that day that a ship can enter the harbour if a minimum depth of 7.5 m of water is required? Give your answer to the nearest minute. 1

(d) A spotlight is in the centre of a rectangular room which measures 32 m by 22m. It is spinning in a clockwise direction at a rate of 25 revolutions/min. Its beam throws a spot which moves along the wall as it spins.



(i) Write the rate of rotation $\frac{d\theta}{dt}$ in radians/sec. 1

(ii) Find an expression for the velocity $\frac{dx}{dt}$ in terms of x at which the spot appears to be moving along the wall from A to B. 2

(iii) Amy stands at point A while Barry stands at point B. They notice that from their different positions in the room the speed at which the spot appears to be moving is also different. From whose position will the spotlight appear to be moving faster? Justify your answer. 2

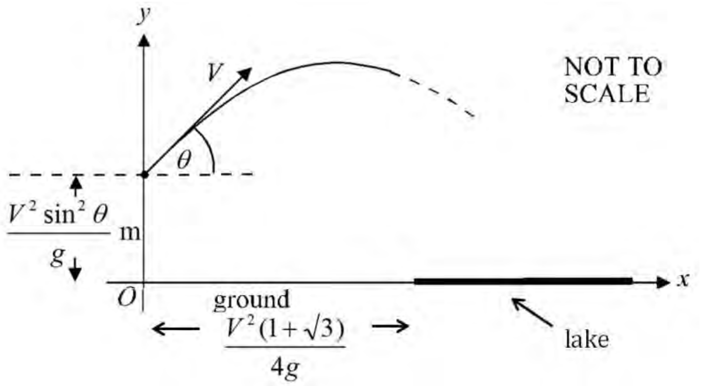
<u>QUESTION 13</u> *(15 Marks)* START A NEW PAGE **Marks**

(a) The probability of winning a certain game is $\frac{1}{3}$. How many times should the game be played so that the probability of winning 4 times is 60 times the probability of winning 6 times? 3

(b) (i) A particle moves in a straight line with acceleration $\frac{dv}{dt} = -3 + 9x$. Initially, the particle is at $x = -1$ with a velocity of $4\ ms^{-1}$. Find x as a function of t. 4

(ii) Describe the motion of the particle when $x = 3$. 1

(c)



An archer stands at the edge of a cliff and shoots an arrow at a constant velocity of $V\ ms^{-1}$ and at an angle of θ to the horizontal, where $0° < \theta < 90°$. The arrow that he shoots is released from a point $\frac{V^2 \sin^2 \theta}{g}$ m vertically above the ground. At ground level, $\frac{V^2(1+\sqrt{3})}{4g}$ m away horizontally from the point of projection is a lake that is $\frac{V^2}{2g}$m wide. The position of the arrow at time t seconds after it is projected is given by the equations:

$$x = Vt\cos\theta\ ; \qquad \text{and } y = -\frac{gt^2}{2} + Vt\sin\theta + \frac{V^2 \sin^2 \theta}{g}$$

(i) Show that the Cartesian equation of the path of the arrow is given by 1

$$y = \frac{-gx^2 \sec^2 \theta}{2V^2} + x\tan\theta + \frac{V^2 \sin^2 \theta}{g}$$

(ii) Show that the horizontal range of the arrow on the ground is given by 2

$$x = \frac{V^2\left(1 + \sqrt{3}\right)\sin 2\theta}{2g}$$

(iii) Find the values of θ for which the arrow will **not** land in the lake or on the edge of the lake. 4

QUESTION 14 *(15 Marks)* START A NEW PAGE **Marks**

(a) (i) Explain why for every positive integer $n, n(n+1)$ is even (no formal proof required). 1

(ii) Hence, using the Principle of Mathematical Induction, prove that for every integer $n \geq 2, n^3 - n$ is a multiple of 6. 3

(b) Find the volume of the solid formed when the region enclosed entirely by the curves $y = \sin x$ and $y = \sin 2x$ over the domain $0 \leq x \leq \frac{\pi}{2}$ is rotated about the x-axis. 4

(c) A, B and C are three points on the circumference of a circle. CB is produced to meet the tangent from A at T. If M is the midpoint of BC, prove that $\angle AOT = \angle AMT$. 3

(d) The number of ways of arranging n students in a row such that no two boys sit together and no two girls sit together is m (m > 100). If one more student is added, the number of ways of arranging the students as above increases by 200%.

(i) Explain why the difference between the number of boys and the number of girls cannot be more than 1. 1

(ii) Show that n cannot be odd. 2

(iii) Hence, or otherwise, find the value of n. 1

END OF PAPER

» Section I

1 mk for each question.

1. A
2. A
3. B
4. D
5. B
6. D
7. C
8. D
9. B
10. A

Suggested Solutions	Marks	Marker's Comments
a) First curve: $\frac{dy}{dx} = 12x^2 + 3 = 12\left(\frac{1}{2}\right)^2 + 3$ at P $= \underline{\underline{6}}$ Second curve: $\frac{dy}{dx} = -\frac{2}{2x} = -\frac{1}{(\frac{1}{2})}$ at P $= \underline{\underline{-2}}$	½, ½	Surprising number did not get the formula exactly correct for acute angle.
Acute angle between lines given by $\tan\theta = \left\lvert \frac{m_1 - m_2}{1 + m_1 m_2} \right\rvert$	1	
$= \left\lvert \frac{6+2}{1-11} \right\rvert = \frac{8}{11}$		½ deducted for 36° 1′
$\therefore \theta = \underline{\underline{36^\circ 2'}}$ (nearest minute)	1	
b) i) Range $\{y : y \in \mathbb{R},\ 0 \le y < \frac{\pi}{2}\}$	1	If the 0 was not given, no mark. ½ off for inequality error at either end.
ii) Let $u = \frac{\sqrt{x^2-4}}{2}$, $y = \tan^{-1} u$ $\frac{du}{dx} = \frac{\frac{1}{2}(x^2-4)^{-\frac{1}{2}} 2x}{2} = \underline{\underline{\frac{x}{2\sqrt{x^2-4}}}}$ $\frac{dy}{du} = \frac{1}{1+u^2} = \frac{1}{1 + \left(\frac{x^2-4}{4}\right)} = \frac{4}{4 + x^2 - 4} = \underline{\underline{4/x^2}}$	1, 1	Not many used the full chain rule 1 each for the main derivatives 1 for some simplification
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (Chain Rule) $= \frac{x}{2\sqrt{x^2-4}} \times \frac{4}{x^2}$ $= \underline{\underline{\frac{2}{x\sqrt{x^2-4}}}}$		

Suggested Solutions	Marks	Marker's Comments
c) i) To show $C = 110 - Ae^{-kt}$ is a solution of $\frac{dC}{dt} = k(110 - C)$. By substitution: $LHS = \frac{dC}{dt} = (-k)(-Ae^{-kt}) = \underline{kAe^{-kt}}$ $RHS = k(110-(110-Ae^{-kt})) = k(Ae^{-kt}) = \underline{kAe^{-kt}}$ $LHS = RHS$ So $\underline{\underline{C = 110 - Ae^{-kt}}}$ is a solution.	1	Fairly easy mark but some structure is required.
ii) At $t = 0$, $C = 15$ $\therefore 15 = 110 - Ae^{0}$ $\therefore \underline{\underline{A = 95}}$	½	½ each for getting value of A and for correctly deciding 99 was the target value
At $t = 19$ $C = 20$ $\therefore 20 = 110 - 95e^{-19k}$ $e^{-19k} = \frac{90}{95} = \frac{18}{19}$ $\therefore k = \underline{\underline{-\frac{1}{19}\ln\left(\frac{18}{19}\right)}} = \underline{\underline{\frac{1}{19}\ln\left(\frac{19}{18}\right)}}$	1	1 mark for k in exact form
Find t when $C = 0.9 \times 110 = \underline{\underline{99}}$	½	
Solve $99 = 110 - 95e^{-kt}$ $e^{-kt} = \frac{11}{95}$ $t = -\frac{1}{k}\ln\left(\frac{11}{95}\right)$ $= \frac{19\ln(95/11)}{\ln(19/18)} = 757.64\ldots$		½ mark if final answer not rounded correctly.
Back to 90% after $\underline{\underline{758 \text{ hrs.}}}$ (nearest hour)	1	

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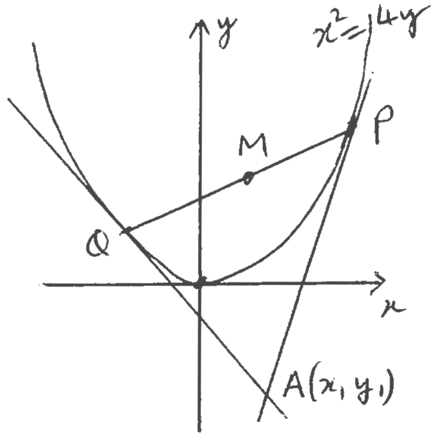
Suggested Solutions	Marks	Marker's Comments

d)

i) P and Q are where chord crosses parabola.



Solve these

$$x^2 = 4y$$
$$xx_1 = 2(y + y_1)$$

Substitute

$$xx_1 = 2\left(\frac{x^2}{4} + y_1\right)$$
$$xx_1 = \frac{x^2}{2} + 2y_1$$
$$x^2 - 2x_1 x + 4y_1 = 0$$

The 2 solutions of this are the x values of P and Q. The x co-ord of M will be the average of these values

i.e. "$\frac{\alpha+\beta}{2}$" where "$\alpha+\beta$" = "$-b/a$" $= 2x_1$

$$\therefore x_M = \tfrac{1}{2}(2x_1) = x_1$$ — Marks: 1

Substitute this into equation of chord:

$$x_1^2 = 2(y_M + y_1)$$
$$x_1^2/2 = y_M + y_1$$
$$y_M = \frac{x_1^2}{2} - y_1$$ — Marks: 1

$\therefore$ M is the point $\left(x_1, \frac{x_1^2}{2} - y_1\right)$

Marker's Comments: Essentially one mark for each value.

Suggested Solutions	Marks	Marker's Comments
ii) Since A moves along line, we have relationship $3x_1 - y_1 - 1 = 0$ $y_1 = 3x_1 - 1$ M is given by $x = x_1$ $y = \frac{x_1^2}{2} - y_1 = \frac{x_1^2}{2} - (3x_1 - 1)$ i.e. $2y = x^2 - 6x + 2$ $2y = (x-3)^2 - 7$ $\underline{2(y + 7/2) = (x-3)^2}$ which is of the correct form.	2.	There was some carelessness with algebra at the end. Most attempts knew what they were doing.

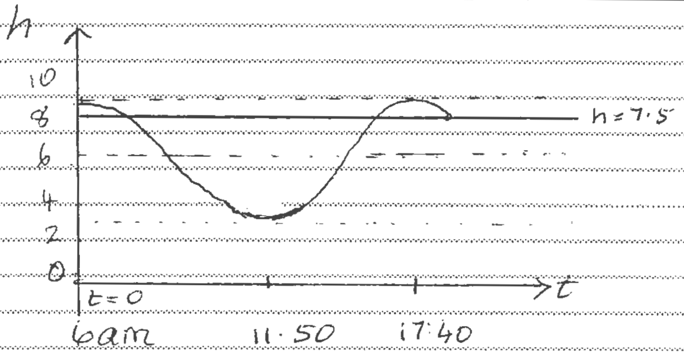
MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
a) $\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx$	(3)	
$x = a\sin u$ $\frac{dx}{du} = a\cos u \quad (dx = a\cos u\,du)$		(1) correct substitution
$= \int \frac{a^2\sin^2 u \cdot a\cos u\,du}{(a^2 - a^2\sin^2 u)^{3/2}}$		
$= \int \frac{a^3\sin^2 u\cos u\,du}{a^3(\cos^3 u)}$		
$= \int \frac{\sin^2 u}{\cos^2 u}\,du$		Correct simplification of Trig to $\tan^2 u$
$= \int \tan^2 u\,du$		(1)
$= \int (\sec^2 u - 1)\,du$ $= \tan u - u + C$		(1) complete integration to correct solution
$= \tan\left[\sin^{-1} x/a\right] - \sin^{-1}\left[\frac{x}{a}\right] + C$		
Alternatively $\sin u = \frac{x}{a} \quad \cos u = \sqrt{1-\frac{x^2}{a^2}}$ for $0 < u < \pi/2$		No loss for ignoring "+c"
$\tan u = \frac{\frac{x}{a}}{\frac{a}{a}\sqrt{a^2-x^2}}$		
$\therefore \int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$		
b) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$	(2)	(1) consideration of ${}^nC_0 x^0$ and its derivative
$= {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 \cdots {}^nC_r x^r \cdots {}^nC_n x^n$		
Differentiate both sides $n(1+x)^{n-1} = 0 + {}^nC_1 + 2\,{}^nC_2 x \cdots r\,{}^nC_r x^{r-1} \cdots n\,{}^nC_n x^{n-1}$		(1) derivative of $(1+x)^n$ and
sub $x = 1$ $n(2)^{n-1} = 0\,{}^nC_0 + {}^nC_1 + 2\,{}^nC_2 \cdots r\,{}^nC_r \cdots n\,{}^nC_n$ as ${}^nC_0 = 1$		sub $x=1$ into both sides
$n(2)^{n-1} = \sum_{r=0}^{n} r \cdot {}^nC_r$		

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments

c) (i)



$t=0$ at 6am

Marks: 3

period $= 2 \times (5 \text{ hrs } 50 \text{ min})$
$= 11 \text{ hrs } 40 \text{ min}$
$= \frac{35}{3}$ hrs

$T = \frac{2\pi}{n}$ $\quad n = \frac{2\pi}{T} = \frac{2\pi}{35/3}$

$n = \frac{6\pi}{35}$

FROM GRAPH

$h = 6.5 + 3.5\cos\left(\frac{6\pi}{35}t + \alpha\right)$ $\quad \alpha = 0$

$= 6.5 + 3.5\cos\left(\frac{6\pi}{35}t\right)$

as $t=0$ is at 6am then 2am $= -4$

$h = 6.5 + 3.5\left(\frac{6\pi}{35} \times -4\right)$

$= 4.57186$

depth $= 4.57$ (nearest cm)

Marker's comments: (1) correct period. (1) correct equation for depth (various solutions). $t=0$ time should be given. (1) correct solution to depth.

(ii) when $h = 7.5$

Marks: 1

$7.5 = 6.5 + 3.5\cos\left(\frac{6\pi}{35}t\right)$

$\frac{1}{3.5} = \cos\left(\frac{6\pi}{35}t\right)$

$\frac{6\pi}{35}t = 2k\pi \pm \cos^{-1}\left(\frac{2}{7}\right)$ $\quad k \in \mathbb{Z}$

$t = \frac{35}{6\pi}\left[2k\pi \pm \cos^{-1}\left(\frac{2}{7}\right)\right]$

Actual time $= \frac{35}{6\pi}\left[2k\pi \pm \cos^{-1}\left(\frac{2}{7}\right)\right] + 6$

Marker's comments: (1) correct substitution and general solution. No loss of mark if (+ 6) not included

(iii) Latest time before midnight from graph is when $k=1$ and $+$ sign

$t = \frac{35}{6\pi}\left[2\pi + \cos^{-1}\left(\frac{2}{7}\right)\right] + 6$

$= 20.04532$

Time $=$ 8:03 pm

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MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments

d)

(i) $25 \text{ rev/min} = \dfrac{25 \times 2\pi}{60}$ — **1** — ① correct answer.

$$\frac{d\theta}{dt} = \frac{5\pi}{6} \text{ rad/sec.}$$

(ii) $\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$ — **2**

$$\tan\theta = \frac{x}{11} \quad \therefore\ x = 11\tan\theta.$$

$$\frac{d\theta}{dx} = 11\sec^2\theta$$ — ① $\dfrac{d\theta}{dx}$ or $\dfrac{dx}{d\theta}$.

$$= 11(\tan^2\theta + 1)$$

$$= 11\left(\frac{x^2}{121} + 1\right)$$

$$= \frac{x^2 + 121}{11}$$

$$\therefore \frac{dx}{dt} = \frac{11}{x^2 + 121}$$

$$\frac{dx}{dt} = \left(\frac{x^2 + 121}{11}\right) \times \frac{5\pi}{6}$$ — ① correct answer in terms of x

$$= \frac{5\pi(x^2 + 121)}{66}$$

Alternatively $\left\{ \theta = \tan^{-1}\left(\dfrac{x}{11}\right) \quad \dfrac{d\theta}{dx} = \dfrac{11}{x^2 + 11^2} \right\}$

(iii) Many alternatives — **2** — No marks for just answer.

① At A $x = 0$ $\quad \dfrac{dx}{dt} = \dfrac{55\pi}{6} = 28.8$

B $x = 16$ $\quad \dfrac{dx}{dt} = \dfrac{55\pi}{6}\left(\dfrac{121 + 16^2}{11}\right) \doteqdot 89.7$

OR — Evidence or explanation must be included

② $\dfrac{dx}{dt} = \dfrac{5\pi}{66}(x^2 + 121)$.

as x increases $0 \le x \le 16$.

$\dfrac{dx}{dt}$ increases $\therefore$ faster at B than A

OR

③ $\dfrac{dx}{dt} = \dfrac{55\pi}{6}(\sec^2\theta)$.

as θ increases $0 \le \theta \le \tan^{-1}\dfrac{16}{11}$

$\dfrac{dx}{dt}$ increases $\therefore$ faster at B than A

④ $\dfrac{d^2x}{dt^2} = \dfrac{5\pi}{33}x$ $\quad$ as $x > 0$ $\quad \dfrac{dx}{dt}$ increases

$\therefore$ faster at B than A

⑤ B is further from light than A

$\therefore$ sweeps out greater distance in same time

as $\dfrac{d\theta}{dt}$ is constant. $\therefore$ Faster at B than A

Q13.

(a) $\left(\frac{1}{3} + \frac{2}{3}\right)^n$

1 $\quad {}^nC_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{n-4} = 60\, {}^nC_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^{n-6}$

$$\frac{n!}{4!(n-4)!} \cdot \left(\frac{2}{3}\right)^2 = 60 \times \frac{n!}{6!(n-6)!} \times \left(\frac{1}{3}\right)^2$$

$$\frac{(n-6)!}{(n-4)!} = 60 \times \frac{4!}{6!} \times \frac{1}{4}$$

$$\frac{1}{(n-4)(n-5)} = \frac{15 \times 1}{6 \times 5}$$

$$\frac{1}{(n-4)(n-5)} = \frac{1}{2}$$

1 $\quad \therefore (n-4)(n-5) = 2$

$$n^2 - 9n + 20 = 2$$

$$n^2 - 9n + 18 = 0$$

$$(n-3)(n-6) = 0$$

$$n = 3 \quad \text{or} \quad n = 6$$

1 $\quad$ but $n \geq 6 \quad \therefore n = 6$ only

* If students forgot the 60 or the ${}^nC_r$ they lost one full mark.
* no conclusion, lost ½ mk.

CORRECT ANS. $\downarrow$

13(a)

$$\frac{dv}{dt} = -3 + 9x$$

$$v\frac{dv}{dx} = -3 + 9x$$

$$\int v\,dv = \int -3 + 9x\,dx$$

$$\frac{v^2}{2} = -3x + \frac{9x^2}{2} + A$$

$$v^2 = -6x + 9x^2 + B.$$

When $x = -1$, $v = 4$

$$\therefore 16 = 6 + 9 + B$$

$$\therefore B = 1$$

$$\therefore v^2 = 9x^2 - 6x + 1$$

$$v^2 = (3x-1)^2$$

$$v = \pm(3x-1)$$

But $v = 4$ when $x = -1$ $\therefore$ − sign

$$v = 1 - 3x$$

$$\frac{dx}{dt} = 1 - 3x$$

$$\int \frac{dx}{1-3x} = \int dt$$

$$-\frac{1}{3}\ln(1-3x) = t + C$$

When $t = 0$, $x = -1$ $\therefore C = -\frac{1}{3}\ln 4$

$$\therefore t = \frac{1}{3}\ln\frac{4}{1-3x}$$

$$3t = \ln\frac{4}{1-3x}$$

$$\frac{1-3x}{4} = e^{-3t}$$

$$x = \frac{1-4e^{-3t}}{3}$$

ERROR IF FIRST CONSTANT OF INTEGRATION NEGLECTED OR PUT EQUAL TO 0.

ERROR $B = 0$!! $\downarrow$

$$v^2 = 9\left(x^2 - \frac{2x}{3}\right)$$

$$v^2 = 9\left(\left(x - \tfrac{1}{3}\right)^2 - \tfrac{1}{9}\right)$$

$$v = \pm 3.\sqrt{\left(x-\tfrac{1}{3}\right)^2 - \tfrac{1}{9}}$$

* Another error here if a sign is assumed - otherwise they should get an error.

If $v = \frac{dx}{dt} = 3\sqrt{\left(x-\tfrac{1}{3}\right)^2 - \tfrac{1}{9}}$

$$\int \frac{dx}{\sqrt{\left(x-\tfrac{1}{3}\right)^2 - \tfrac{1}{9}}} = \int 3\,dt$$

$$\sin^{-1}\left(3\left(x-\tfrac{1}{3}\right)\right) = 3t + C$$

$$\sin^{-1}(3x-1) = 3t + C$$

* Another problem $\sin^{-1}(-4)$ does not exist. So a fish should be smelt.

ERROR $C = 0$

$$3x - 1 = \sin 3t$$

$$x = \frac{1 + \sin 3t}{3}.$$

<u>Even to get to this answer has required 3 errors.</u>

(b) (i) $\frac{dv}{dt} = -3 + 9x$ when $t=0, x=-1, v=4$

$a = v\frac{dv}{dx}$ $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = a$

$a = -3 + 9x$

$$\int (-3 + 9x)\,dx = \frac{1}{2}v^2$$

$$-3x + \frac{9}{2}x^2 + c = \frac{1}{2}v^2$$

$$-6x + 9x^2 + 2c = v^2$$

when $x = -1, v = 4$

$$6 + 9 + 2c = 16$$

$$2c = 1 \qquad (1)$$

$$\therefore 9x^2 - 6x + 1 = v^2$$

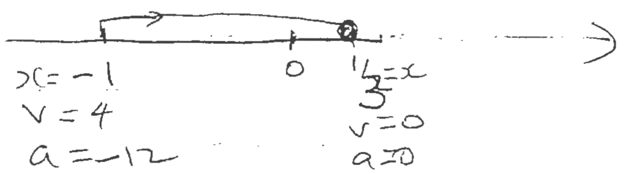
$$v^2 = (3x-1)^2$$

$$v = \pm(3x-1)$$

$3x - 1 > 0$

$3x > 1$

$x > \frac{1}{3}$



$$\therefore v = -3x + 1 \qquad (1)$$

$$\frac{dx}{dt} = -3x + 1$$

$$\frac{dt}{dx} = \frac{1}{1-3x}$$

$$t = -\frac{1}{3}\ln(1-3x) + K$$

when $t = 0, x = -1$

$$0 = -\frac{1}{3}\ln(4) + K$$

$$K = \frac{1}{3}\ln(4)$$

$$\therefore t = -\frac{1}{3}\ln(1-3x) + \frac{1}{3}\ln 4 \qquad (1)$$

$$3t = \ln 4 - \ln(1-3x)$$

$$3t = \ln\left(\frac{4}{1-3x}\right)$$

$$e^{3t} = \frac{4}{1-3x}$$

$$(1-3x) = \frac{4}{e^{3t}}$$

$$1 - \frac{4}{e^{3t}} = 3x$$

$$x = \frac{1}{3}\left(1 - \frac{4}{e^{3t}}\right) \qquad (1)$$

(c) (i) $x = vt\cos\theta$ $\quad y = \frac{-gt^2}{2} + vt\sin\theta + \frac{v^2\sin^2\theta}{g}$

from x ; $t = \frac{x}{v\cos\theta}$ sub. into y

$\therefore y = -\frac{g}{2}\left(\frac{x}{v\cos\theta}\right)^2 + v\sin\theta\left(\frac{x}{v\cos\theta}\right) + \frac{v^2\sin^2\theta}{g}$ *

1

$= \frac{-g}{2}\frac{x^2}{v^2\cos^2\theta} + \frac{vx\tan\theta}{v} + \frac{v^2\sin^2\theta}{g}$ *

$= -\frac{gx^2\sec^2\theta}{2v^2} + x\tan\theta + \frac{v^2\sin^2\theta}{g}$ or

(ii) when $y = 0$?? $t = ?$ $x = ??$

method <u>one</u>

$0 = \frac{-g}{2}t^2 + vt\sin\theta + \frac{v^2\sin^2\theta}{g}$

$t = \frac{-v\sin\theta \pm \sqrt{v^2\sin^2\theta - 4\left(-\frac{g}{2}\right)\left(\frac{v^2\sin^2\theta}{g}\right)}}{2\times -g/2}$

1/2

$= \frac{-v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2v^2\sin^2\theta}}{-g}$

$= \frac{-v\sin\theta \pm \sqrt{3}v\sin\theta}{-g}$

$= \frac{v\sin\theta + \sqrt{3}v\sin\theta}{g}$ or $\frac{v\sin\theta - \sqrt{3}v\sin\theta}{g}$ $\frac{1}{2}$

$\therefore t = \frac{\sin\theta v(1+\sqrt{3})}{g}$ $\qquad \frac{v\sin\theta(1-\sqrt{3})}{g}$

but $t \geq 0$ $\therefore t = \frac{\sin\theta v(1+\sqrt{3})}{g}$ only 1/2

$\therefore x = v\cos\theta \times \frac{\sin\theta . v(1+\sqrt{3})}{g} = \frac{v^2\sin 2\theta(1+\sqrt{3})}{2g}$ 1/2

(ii) when $y=0$

<u>method two</u> when $y=0$

$$0 = -\frac{g\sec^2\theta}{2v^2}x^2 + x.\tan\theta + \frac{v^2\sin^2\theta}{g}$$

$$\therefore x = \frac{-\tan\theta \pm \sqrt{\tan^2\theta - 4\left(-\frac{g\sec^2\theta}{2v^2}\right)\left(\frac{v^2\sin^2\theta}{g}\right)}}{2\left(-\frac{g\sec^2\theta}{2v^2}\right)}$$

$$x = \frac{-\tan\theta \pm \sqrt{\tan^2\theta + 2\sec^2\theta.\sin^2\theta}}{-\frac{g\sec^2\theta}{v^2}}$$

1

$$= \left(\frac{-\tan\theta \pm \sqrt{\tan^2\theta + 2\tan^2\theta}}{-g\sec^2\theta}\right)v^2$$

$$= \frac{(-\tan\theta \pm \sqrt{3}\tan\theta)v^2}{-g\sec^2\theta}$$

$$= \frac{\tan\theta(-1 \pm \sqrt{3})v^2}{-g\sec^2\theta}$$

but $x > 0$ as its a length

1

$$\therefore x = \frac{\tan\theta(1+\sqrt{3})v^2}{g\sec^2\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} \times \frac{\cos^2\theta}{1}(1+\sqrt{3})v^2}{g}$$

$$= \frac{\sin\theta\cos\theta(1+\sqrt{3})v^2}{g}$$

$$= \frac{\sin 2\theta(1+\sqrt{3})v^2}{2g}$$

QED.

(iii) to NOT land in the lake

$$① \quad x < \frac{v^2(1+\sqrt{3})}{4g} \quad \text{or} \quad ② \quad x > \frac{v^2(1+\sqrt{3})}{4g} + \frac{v^2}{2g}$$

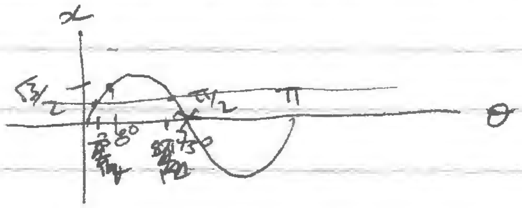
① $$x = \frac{v^2(1+\sqrt{3})\sin 2\theta}{2g} \quad \text{from ex(ii)}$$

$$\therefore \frac{\cancel{v^2}\cancel{(1+\sqrt{3})}\sin 2\theta}{2\cancel{g}} < \frac{\cancel{v^2}\cancel{(1+\sqrt{3})}}{4\cancel{g}}$$

$$2\sin 2\theta < 1$$

$$\sin 2\theta < \tfrac{1}{2}$$

1



$$0^\circ < \theta < 15^\circ \quad \text{or} \quad 75^\circ < \theta < 90^\circ$$

1

or ② $$\frac{\cancel{v^2}(1+\sqrt{3})\sin 2\theta}{2\cancel{g}} > \frac{\cancel{v^2}(1+\sqrt{3})}{4\cancel{g}} + \frac{2\cancel{v^2}}{4\cancel{g}}$$

$$2(1+\sqrt{3})\sin 2\theta > 1+\sqrt{3}+2$$

$$(1+\sqrt{3})\sin 2\theta > \frac{3}{2} + \frac{\sqrt{3}}{2}$$

$$(1+\sqrt{3})\sin 2\theta > \frac{3+\sqrt{3}}{2}$$

1

$$\sin 2\theta > \frac{3+\sqrt{3}}{2(1+\sqrt{3})} \qquad \sin 2\theta > \frac{\sqrt{3}(\sqrt{3}+1)}{2(1+\sqrt{3})}$$

$$\sin 2\theta > \frac{\sqrt{3}}{2}$$

$$30^\circ < \theta < 60^\circ$$

1

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SOLUTION	MARKS	COMMENTS
QUESTION 14		
a) (i)		
If n = even, then $n+1$ = odd		Must show
$\Rightarrow n(n+1)$ = even × odd = even		both cases
	1	for full marks
If n = odd, then $n+1$ = even		
$\Rightarrow n(n+1)$ = odd × even = even		
(ii) Let $S_n = n^3 - n \mid 6$		
Prove S_2 is true i.e. $n=2$		
When $n=2$ $S_2 = 2^3 - 2 = 6$ which		
is divisible by 6		
$\therefore$ true for $n=2$		
	1	For initial setup
Assume true for $n = k \geq 2, \in Z^+$		and assumption
i.e $k^3 - k = 6Q, \quad Q \in Z$		statement
Prove S_{k+1} is true		
i.e. $(k+1)^3 - (k+1) = 6P, \; P \in Z$		
LHS $= (k+1)\left[(k+1)^2 - 1\right]$		
$= (k+1)(k^2+2k)$		
$= (k+1)k(k+2)$		correct
$= k^3 + 3k^2 + 2k$	1	Simplification
$= (k^3 - k) + 3k^2 + 3k$		of LHS of S_n

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$= 6Q + 3k(k+1)$ ... by assumption		
But from (1) $k(k+1)$ is even.		
Therefore $3k(k+1)$ is divisible		
by 6 as $3 \times \text{even} \mid 6$		
$\therefore LHS = 6Q + 3 \times 2M$		
$= 6Q + 6M, \; M \in \mathbb{Z}$		For correct
$= 6(Q+M)$	1	verification of
where $Q+M \in \mathbb{Z}$		LHS being
$\therefore (k+1)^3 - (k+1)$ is divisible by 6		divisible by 6
i.e. $(k+1)^3 - (k+1) \mid 6$		
$\therefore$ By the PMI, $n^3 - n \mid 6$		

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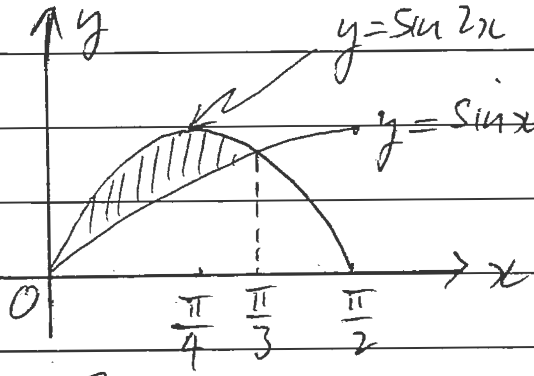
h) 		Note: POI: $\sin 2x = \sin x$ $2\sin x\cos x = \sin x$ $\sin x(2\cos x - 1) = 0$ $\sin x = 0$ or $\cos x = \frac{1}{2}$ $x = 0$ or $x = \pi/3$ for $0 \le x \le \pi/2$
$V = \pi\int_0^{\pi/3}(\sin^2 2x - \sin^2 x)\,dx$		
$= \pi\int_0^{\pi/3}\left[\frac{1}{2}(1-\cos 4x) - \frac{1}{2}(1-\cos 2x)\right]dx$	1	Correct setup for V ie $\int$ square
$= \frac{\pi}{2}\int_0^{\pi/3}(\cos 2x - \cos 4x)\,dx$	1	$\int \cos 2x - \cos 4x$ (Use of double ∠)
$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\pi/3}$	1	Correct integration
$= \frac{\pi}{8}\left[2\times\sin\frac{2\pi}{3} - \sin\frac{4\pi}{3} - 0 - 0\right]$		
$= \frac{\pi}{8}\left(2\times\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right)$		
$= \frac{3\pi\sqrt{3}}{16}\ u^3$	1	Correct evaluation of definite integral.

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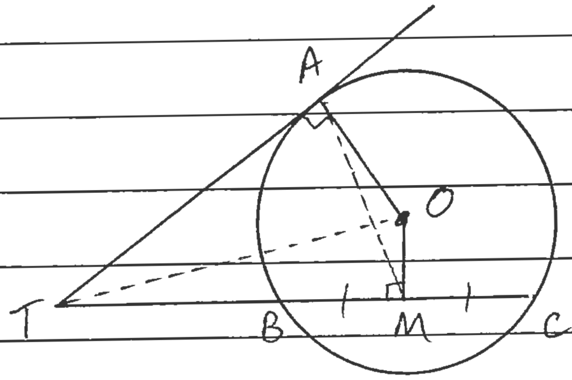
c) 		
$\angle OMT = 90^\circ$ ... (line from centre to midpoint of chord is perpendicular to it)	1	for 2 right angles
$\angle OAT = 90^\circ$ ... (radius $\perp$ tangent at point of contact		
$\therefore$ In quadrilateral AOMT $\angle OMT + \angle OAT = 90^\circ + 90^\circ$ $= 180^\circ$	1	for correct choice of supplementary angles or exterior and sum of interior angles
$\therefore$ quadrilateral AOMT is cyclic ... (opposite angles are supplementary)		
In cyclic quadrilateral AOMT $\angle AOT = \angle AMT$ ... (angles in same segment subtended by same chord/arc are equal)	1	for correct reason for final result.

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d)		
(i) The condition that no 2		Easily obtained
girls or boys sit together		
implies that they be seated	1	
B, G, B, G, etc alternately		
(ii) Let		Poorly answered
x = number of boys, then		due to either
$x+1$ = number of girls		ambiguity of
So total number of arrangements		question or
is BGBG...		poor intepretation
or GBGB...		
i.e $2x!\,(x+1)!$		Generally,
i.e. $2x!(x+1)!$		proof by
i.e. $m = 2x!\,(x+1)!$		contradiction
		was accepted
If 1 is added, then the		or any
number of arrangements is		sensible
$2 \times (x+1)!\,(x+1)!$		argument.
But an increase of 200%		
$\Rightarrow m \to 3m$		For setting up
i.e. $3 \cdot x!\,(x+1)! = 2\,(x+1)!\,(x+1)!$		argument
$3x! = 2(x+1)!$	1	
$\Rightarrow x = \frac{1}{2}$ (How?)		

which is impossible!		For justification
Hence a contradiction.	1	
$\therefore$ n cannot be odd.		
(iii) Initial number of arrangements		
when n is even is		
$2x!x!$		
But increases three fold (200%)		
$\therefore$ final number of		
arrangements is $3\times 2x!x!$		for final
when 1 more is added	1	answer of 10
i.e $x!(x+1)! = 6x!x!$		
i.e. $(x+1)! = 6x!$		
$\Rightarrow (x+1)x! = 6$		
$x = 5$		
of one gender		
Hence $n = 2\times 5$		
$= 10$ people		